LR and L+R systems

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Abstract
We consider coupled nonholonomic LR systems on the product of Lie groups. As examples, we study n-dimensional variants of the spherical support system and the rubber Chaplygin sphere. For a special choice of the inertia operator, it is proved that the rubber Chaplygin sphere, after reduction and a time reparametrization becomes an integrable Hamiltonian system on the $(n-1)$-dimensional sphere. Also, we showed that an arbitrary L+R system introduced by Fedorov can be seen as a reduced system of an appropriate coupled LR system.

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1. Introduction

In this paper we study nonholonomic geodesic flows on the direct product of Lie groups with specially chosen right-invariant constraints and left-invariant metrics.

Let Q be an n-dimensional Riemannian manifold Q with a nondegenerate metric $\kappa(\cdot, \cdot)$ and let $\mathcal{D}$ be a nonintegrable $(n-k)$-dimensional distribution on the tangent bundle TQ. A smooth path $q(t) \in Q, t \in \Delta$ is called *admissible* (or allowed by constraints) if the velocity $\dot{q}(t)$ belongs to $\mathcal{D}_{q(t)}$ for all $t \in \Delta$. Let $q = (q_1, \ldots, q_n)$ be some local coordinates on Q in which the constraints are written in the form

$$\left(\alpha_q^j, \dot{q}\right) = \sum_{i=1}^{n} \alpha_i^j \dot{q}_i = 0, \qquad j = 1, \ldots, k, \tag{1}$$

where α^j are independent 1-forms. The admissible path $q(t)$ is called a *nonholonomic geodesic* if it is satisfies the Lagrange–d'Alembert equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \sum_{i=1}^{k} \lambda_j \alpha^j(q)_i, \qquad i = 1, \ldots, n, \tag{2}$$

where the Lagrange multipliers λ_j are chosen such that the solutions $q(t)$ satisfy constraints (1) and the Lagrangian is given by the kinetic energy $L = \frac{1}{2}\kappa(\dot{q}, \dot{q}) = \frac{1}{2}\sum_{ij}\kappa_{ij}\dot{q}_i\dot{q}_j$. After the Legendre transformation $p_i = \partial L/\dot{q}_i = \sum_j \kappa_{ij}\dot{q}_j, i = 1, \ldots, n$, one can also write the Lagrange–d'Alambert equations as a first-order system on the cotangent bundle T^*Q. As for the Hamiltonian systems, the Lagrangian $L(q, \dot{q})$ (or the Hamiltonian $H(q, p) = \frac{1}{2}\sum_{ij}\kappa^{ij}p_ip_j$ in the cotangent representation of the flow) is always the first integral of the system.

Suppose that a Lie group K acts by isometries on (Q, κ) (the Lagrangian L is K-invariant) and let ξ_Q be the vector field on Q associated with the action of the one-parameter subgroup $\exp(t\xi), \xi \in \mathfrak{k} = T_{\mathrm{Id}}K$. The following version of the Noether theorem holds (see [1, 2]): if ξ_Q is a section of the distribution $\mathcal{D}$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}, \xi_Q\right) = 0. \tag{3}$$

On the other hand, let ξ_Q be transversal to $\mathcal{D}$, for all $\xi \in \mathfrak{k}$. In addition, suppose that Q has a principal bundle structure $\pi : Q \to Q/K$ and that $\mathcal{D}$ is the collection of horizontal spaces of a principal connection. Then the nonholonomic geodesic flow defined by $(Q, \kappa, \mathcal{D})$ is called a *K-Chaplygin system*. The system (2) is K-invariant and reduces to the tangent bundle $T(Q/K) = \mathcal{D}/K$ (for the details see [2, 8, 11, 26]).

Equations (2) are not Hamiltonian. However, in some cases they have a rather strong property—an invariant measure (e.g., see [1, 4, 27]). Within the class of K-Chaplygin systems, the existence of an invariant measure is closely related with their reduction to a Hamiltonian form after an appropriate time rescaling $\mathrm{d}\tau = \mathcal{N}\,\mathrm{d}t$ (see [8, 10, 11, 18, 29]).

Veselov and Veselova [30, 31] constructed nonholonomic systems on unimodular Lie groups with right-invariant nonintegrable constraints and left-invariant metrics, so-called *LR systems*, and showed that they always possess an invariant measure. Similar integrable nonholonomic problems on Lie groups, with left- and right-invariant constraints, are studied in [3, 17, 19, 21, 22]. Recently, a nontrivial example of a nonholonomic LR system, which can be regarded also as a generalized Chaplygin system (*n-dimensional Veselova rigid body problem* [17, 30]) such that the Chaplygin reducibility theorem is applicable for any dimension was given by Fedorov and Jovanović [18].

It appears that LR systems can be viewed as a limit case of certain artificial systems (L+R systems) on the same group, which also possess an invariant measure (see Fedorov [15]). The latter systems do not have a straightforward mechanical or geometric interpretation and arise as a 'distortion' of a geodesic flow on G whose kinetic energy is given by sum of a left- and a right-invariant metric. On the other hand, we shall prove that an arbitrary L+R system on G can be obtained as a reduction of a *coupled* nonholonomic LR system defined on the direct product $G \times G$.

In Schneider [28] a class of nonholonomic systems defined on a semi-direct product of a group G and a vector space V that are reducible to L+R systems on G is considered. One of the best known examples of integrable nonholonomic systems with an invariant measure is the celebrated Chaplygin sphere which describes a dynamically non-symmetric ball rolling without sliding on a horizontal plane [1, 9]. It is interesting that Chaplygin's sphere as well as its n-dimensional generalization fits within both constructions. In the construction described in [28] the configuration space is the Lie group of Euclidean motion $SE(n)$, that is the *semidirect* product of $SO(n)$ and $\mathbb{R}^n$ [28]. Besides, the Chaplygin sphere is an LR system on the *direct* product $SO(n) \times \mathbb{R}^n$ (e.g., see [16]). This was a starting point in considering the coupled nonholonomic LR systems below.

1.1. Outline and results of the paper

In section 2 we recall the definition and basic properties of LR and L+R systems. We define the coupled LR systems and show that any L+R system can be obtained as a reduction of an appropriate coupled LR system (sections 3 and 4). An example of a coupled LR system on $G \times \mathfrak{g}$ is given, which provides an alternative generalization of the Chaplygin sphere problem (section 4, system (73) in section 6).

In section 5 we study an n-dimensional variant of the spherical support system introduced by Fedorov [13]: the motion of a dynamically nonsymmetric ball $\mathcal{S}$ with the unit radius around its fixed center that touches N arbitrary dynamically symmetric balls whose centers are also fixed, and there is no sliding at the contact points.

Recall that the rubber rolling of the sphere S^2 over some other fixed convex surface in $\mathbb{R}^3$ means that in the addition to the constraint given by the condition that the velocity of the contact point is equal to zero, we have the no-twist condition that rotations about the normal to the surface are forbidden. The rubber rolling of the dynamically non-symmetric sphere over another sphere, considered as a Chaplygin system on the bundle $SO(3) \times S^2 \to S^2$ (where $SO(3)$ acts diagonally on the total space), as well as the Hamiltonization in sphero-conical variables of S^2 is given by Koiller and Ehlers [12]. The integrable cases are found by Borisov and Mamaev [7]. In particular, when the radius of the fixed sphere tends to infinity, we get the rubber rolling of the sphere over the plane (*rubber Chaplygin sphere*). The Chaplygin reducing multiplier for the rubber Chaplygin sphere is given in [11].

By the analogue, we define the n-dimensional rubber spherical support system with additional no-twist conditions at the contact points. It appears that both systems fit into the construction of coupled LR systems. Similarly as for the three-dimensional spherical support system studied in [13], we prove that the three-dimensional rubber spherical support system is integrable (section 5).

Finally, in section 6 we consider the n-dimensional rubber Chaplygin sphere problem describing the rolling without slipping and twisting of an n-dimensional ball on an $(n-1)$-dimensional hyperplane $\mathcal{H}$ in $\mathbb{R}^n$ as coupled LR systems on the direct product $SO(n) \times \mathbb{R}^{n-1}$. It appears that the rubber Chaplygin sphere is a $SO(n-1) \times \mathbb{R}^{n-1}$-Chaplygin system closely related to the n-dimensional nonholonomic Veselova problem, which allows as to prove the existence of the Chaplygin multiplier for a specially chosen inertia operator of the ball. In particular, when $n = 3$, the multiplier exists for any inertia tensor of the ball and reduces to that obtained in [11, 12].

2. Preliminaries

2.1. LR systems

An LR system on a Lie group G is a nonholonomic geodesic flow of a left-invariant metric and right-invariant nonintegrable distribution $\mathcal{D} \subset TG$ (see [30, 31]). Through the paper we suppose that all considered Lie groups G have bi-invariant Riemannian metrics, or equivalently Ad_G-invariant Euclidean scalar products $\langle \cdot, \cdot \rangle$ on corresponding Lie algebras $\mathfrak{g} = T_{\mathrm{Id}}G$. In particular, Lie groups G are unimodular, i.e., $\mathrm{tr}\, \mathrm{ad}_\omega = 0$, for all $\omega \in \mathfrak{g}$. In what follows we shall identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of an invariant scalar product $\langle \cdot, \cdot \rangle$, and TG and T^*G by the bi-invariant metric. For clarity, we shall use the symbol ω for the elements in $\mathfrak{g}$ and the symbol m for the elements in $\mathfrak{g}^* \cong \mathfrak{g}$.

The Lagrangian is defined by $L(g, \dot{g}) = \frac{1}{2}\langle I\omega, \omega \rangle$, where $\omega = g^{-1} \cdot \dot{g}$ is the *angular velocity in the moving frame*. Here $I : \mathfrak{g} \to \mathfrak{g}$ is a symmetric positive definite (with respect

to $\langle \cdot, \cdot \rangle$) operator. The corresponding left-invariant metric will be denoted by $(\cdot, \cdot)_I$. The distribution $\mathcal{D}$ is determined by its restriction $\mathfrak{d}$ to the Lie algebra and it is nonintegrable if and only if $\mathfrak{d}$ is not a subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}$ be the orthogonal complement of $\mathfrak{d}$ with respect to $\langle \cdot, \cdot \rangle$ and let $a_1, \ldots, a_k$ be an orthonormal base of $\mathfrak{h}$. Then the right-invariant constraints can be written as

$$\langle \Omega, \mathfrak{h} \rangle = \langle \omega, \mathfrak{h}^g \rangle = 0, \qquad \mathfrak{h}^g = \mathrm{Ad}_{g^{-1}}(\mathfrak{h}) = g^{-1} \cdot \mathfrak{h} \cdot g,$$

or, equivalently,

$$\langle \alpha_i, \omega \rangle = 0, \qquad \alpha_i = \mathrm{Ad}_{g^{-1}}(a_i), \qquad i = 1, \ldots, k. \tag{4}$$

Here $\Omega = \mathrm{Ad}_g(\omega) = \dot{g} \cdot g^{-1}$ represents *angular velocity in the space*.

Equations (2) in the left trivialization take the form

$$\dot{m} = [m, \omega] + \sum_{i=1}^{k} \lambda_i \alpha_i, \tag{5}$$

$$\dot{g} = g \cdot \omega, \tag{6}$$

where $m = \partial L / \partial \omega = I \omega \in \mathfrak{g}^*$ is the *angular momentum in the body frame*.

The Lagrange multipliers λ_i can be found by differentiating the constraints (4). They are actually defined on the whole phase space T^*G and we can consider the system (5) and (6) on T^*G as well (see [31]). The constraint functions $\langle \alpha_i, \omega \rangle$ are then integrals of the extended system and the nonholonomic geodesic flow is just the restriction of (5) and (6) onto the invariant submanifold (4).

Instead of (5) and (6), one can consider the closed system consisting of (5) and

$$\dot{\alpha}_i = [\alpha_i, \omega], \qquad i = 1, \ldots, k, \tag{7}$$

on the direct product $\mathfrak{g}^{1+k} = \{(m, \alpha^1, \ldots, \alpha^k)\}$. Let $I^{-1}|_{\mathfrak{h}^g} = \mathrm{pr}_{\mathfrak{h}^g} \circ I^{-1} \circ \mathrm{pr}_{\mathfrak{h}^g}$, where $\mathrm{pr}_{\mathfrak{h}^g}$ is the orthogonal projection to $\mathfrak{h}^g$. Then the system (5) and (7) has an invariant measure with density $\mu = \sqrt{\det(I^{-1}|_{\mathfrak{h}^g})} = \sqrt{\det(\langle I^{-1}(\alpha_i), \alpha_j \rangle)}$ (see [31]).

Also, since for $\xi \in \mathfrak{g}$, the associate vector field ξ_G of the left G-action is right invariant and the momentum mapping of the left action equals $M = \mathrm{Ad}_g(m)$ (*angular momentum in the space*), the LR system (5) and (6) has the Noether conservation laws:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathrm{Ad}_g(m), \xi \rangle = 0, \qquad \xi \in \mathfrak{d}. \tag{8}$$

If the linear subspace $\mathfrak{h}$ is the Lie algebra of a subgroup $H \subset G$, then the Lagrangian L and the right-invariant distribution $\mathcal{D}$ are invariant with respect to the left H-action. As a result, the LR system can naturally be regarded as an H-Chaplygin system [18].

2.2. Geodesic flow on G with L+R metric

In addition to the nondegenerate linear operator I defining the left-invariant metric $(\cdot, \cdot)_I$, introduce a constant symmetric linear operator $\Pi^0 : \mathfrak{g} \to \mathfrak{g}$ defining a right-invariant metric $(\cdot, \cdot)_\Pi$ on the n-dimensional compact Lie group G: for any vectors $u, v \in T_g G$ we put $(u, v)_\Pi = \langle ug^{-1}, \Pi^0 vg^{-1} \rangle$. We take the sum of both metrics and consider the corresponding geodesic flow on G described by the Lagrangian

$$L = \tfrac{1}{2} \langle \omega, I \omega \rangle + \tfrac{1}{2} \langle g \omega g^{-1}, \Pi^0 g \omega g^{-1} \rangle = \tfrac{1}{2} \langle \omega, I \omega \rangle + \langle \omega, \Pi^g \omega \rangle,$$

where $\Pi^g = \mathrm{Ad}_{g^{-1}} \Pi^0 \mathrm{Ad}_g$. We can also consider the case when Π^g is not positive definite, but the total inertia operator $\mathcal{B} = I + \Pi^g$ is nondegenerate and positive definite on the whole group G.

The geodesic motion on the group is described by the Euler–Poincaré equations

$$\dot{m} = [m, \omega] + g^{-1}\frac{\partial L}{\partial g}, \qquad m = \frac{\partial L}{\partial \omega} = \mathcal{B}\omega, \tag{9}$$

together with the kinematic equation $\dot{g} = g \cdot \omega$.

In order to find an explicit expression for $g^{-1}(\partial L/\partial g)$, we first note that for any $\xi \in \mathfrak{g}$, $\langle \xi, g^{-1}(\partial L/\partial g)\rangle = v_\xi(L)$, where v_ξ is the left-invariant vector field on G generated by ξ. Since the metric $(\cdot, \cdot)_I$ is left invariant, we have

$$v_\xi(L) = \tfrac{1}{2}v_\xi(\langle \omega, \Pi\omega\rangle) = \tfrac{1}{2}\langle \omega, \Pi\,\mathrm{ad}_\xi\,\omega + \mathrm{ad}_\xi^T\,\Pi\omega\rangle = \langle \Pi\omega, [\xi, \omega]\rangle = \langle \xi, \mathrm{ad}_\omega\,\Pi\omega\rangle.$$

As a result, $g^{-1}(\partial L/\partial g) = \mathrm{ad}_\omega\,\Pi\omega$.

Also, in view of the definition of Π, its evolution is given by the $n \times n$ matrix equation

$$\dot{\Pi} = \Pi\,\mathrm{ad}_\omega + \mathrm{ad}_\omega^T\Pi. \tag{10}$$

Since $\langle \cdot, \cdot \rangle$ is Ad_G invariant scalar product, we have $\mathrm{ad}_\omega^T = -\mathrm{ad}_\omega$, and $\dot{\Pi} = [\Pi, \mathrm{ad}_\omega]$.

Equations (9) and (10) form a closed system on the space $\mathfrak{g} \times \mathbf{Symm}(n)$ with the coordinates ω_i, Π_{ij} $\left(\omega = \sum_i \omega_i e_i, \Pi = \sum_{i \leqslant j} \Pi_{ij} e_i \otimes e_j\right)$, where $e_1, \ldots, e_n$ is an orthonormal base of $\mathfrak{g}$.

2.3. L+R systems

Following Fedorov [15], consider equations (9) modified by rejecting the term $g^{-1}(\partial L/\partial g)$. As a result, we obtain the another system

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}\omega) = [\mathcal{B}\omega, \omega], \qquad \dot{g} = g \cdot \omega, \qquad \mathcal{B} = I + \Pi \tag{11}$$

on TG, or the system

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}\omega) = [\mathcal{B}\omega, \omega], \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Pi = \Pi\,\mathrm{ad}_\omega + \mathrm{ad}_\omega^T\Pi \tag{12}$$

on the space $\mathfrak{g} \times \mathbf{Symm}(n)$. This is generally not a Lagrangian system, and, in contrast to equations (9) and (10), it possesses the 'momentum' integral $\langle \mathcal{B}\omega, \mathcal{B}\omega\rangle$. In view of the structure of the kinetic energy, we shall refer to the system (11) (or (12)) as *L+R system* on G [15].

The L+R system (12) possesses also the kinetic energy integral $\tfrac{1}{2}\langle \omega, \mathcal{B}\omega\rangle$ and an invariant measure (in coordinates ω_i, Π_{ij}) $\mu\,\mathrm{d}\omega_1 \wedge \cdots \wedge \mathrm{d}\omega_n \wedge \mathrm{d}\Pi_{11} \wedge \cdots \wedge \mathrm{d}\Pi_{nn}$ with density $\mu = \sqrt{\det(I + \Pi)}$ (see [14, 15]).

As mentioned above, a nonholonomic LR system on a Lie group G can be obtained as a limit case of a certain L+R system on this group. Indeed, suppose that the operator defining a right-invariant metric on G is *degenerate* and has the form $\Pi = \epsilon(\alpha_1 \otimes \alpha_1 + \cdots + \alpha_k \otimes \alpha_k)$, $k < n, \epsilon = \mathrm{const} > 0$, where, as in (4), $\alpha_1, \ldots, \alpha_k$ are orthonormal right-invariant vector fields $\alpha_i = g^{-1} \cdot a_i \cdot g, a_i = \mathrm{const} \in \mathfrak{g}$. The L+R system (12) on the space $(\omega, \alpha_1, \ldots, \alpha_k)$ can be represented in form

$$I\dot{\omega} = I(I + \Pi)^{-1}[I\omega, \omega], \qquad \dot{\Pi} = \Pi\,\mathrm{ad}_\omega + \mathrm{ad}_\omega^T\Pi. \tag{13}$$

Then the following statement holds (see [15]). As $\epsilon \to \infty$, equations (13) transform to the equations with multipliers (5) and constraints (4), where $m = I\omega$. Also the density $\sqrt{\det \mathcal{B}}/\sqrt{\epsilon}$ of the invariant measure of the L+R system tends to the density of the LR system multiplied by a constant factor. Note that as $\epsilon \to \infty$, the original equations (12) become singular. For this reason, before taking the limit they must be transformed to the form (13).

3. Coupled nonholonomic LR systems

Define a *coupled nonholonomic LR system* on the direct product $G \times G_1$ ($G = G_1$) as a LR system given by the Lagrangian function

$$L = \tfrac{1}{2}\langle I\omega, \omega\rangle + \tfrac{1}{2}D\langle \mathbf{w}, \mathbf{w}\rangle \tag{14}$$

and right-invariant constraints

$$\langle \Omega, \mathfrak{h}_0 \rangle = 0, \tag{15}$$

$$\langle \Omega + \rho_i \mathbf{W}, \mathfrak{h}_i \rangle = 0, \qquad i = 1, \ldots, q, \tag{16}$$

where $\mathfrak{h}_i$, $i = 1, \ldots, q$ are mutually orthogonal linear subspaces of $\mathfrak{g}$.

Here $(\omega, \mathbf{w}) = \left(g^{-1}\dot{g}, g_1^{-1}\dot{g}_1\right)$, is the angular velocity in the body and $(\Omega, \mathbf{W}) = \mathrm{Ad}_{(g, g_1)} = (\mathrm{Ad}_g(\omega), \mathrm{Ad}_{g_1}(\mathbf{w}))$ is the angular velocity in the space, $(g, g_1) \in G \times G_1$. The constant D is greater than zero, while ρ_i, $i = 1, \ldots, q$ are arbitrary non-zero, real parameters.

The Lagrangian (14) in the second variable is right invariant as well. It is convenient to write the equations of motion both in the left trivialization (in variables g and ω) and right trivialization (in variables g_1 and $\mathbf{W}$)

$$T(G \times G_1) \approx G \times G_1 \times \mathfrak{g} \times \mathfrak{g}_1 = \{(g, g_1, \omega, \mathbf{W})\}. \tag{17}$$

Then the right-invariant distribution $\mathcal{D} \subset T(G \times G_1)$ is given by

$$\mathcal{D} = \{(g, g_1, \omega, \mathbf{W}) | \langle \mathrm{Ad}_g(\omega), \mathfrak{h}_0 \rangle = 0, \langle \mathrm{Ad}_g(\omega) + \rho_i \mathbf{W}, \mathfrak{h}_i \rangle = 0, i = 1, \ldots, q \}.$$

Let $\mathfrak{h}_i^g = \mathrm{Ad}_{g^{-1}}(\mathfrak{h}_i) = g^{-1} \cdot \mathfrak{h}_i \cdot g$ and let $\mathrm{pr}_{\mathfrak{h}_i^g} : \mathfrak{g} \to \mathfrak{h}_i^g$ be the orthogonal projections, $i = 0, \ldots, q$.

Proposition 3.1. *The admissible path $(g(t), g_1(t), \omega(t), \mathbf{W}(t))$ is a motion of the nonholonomic LR system (14)–(16) if it satisfies equations*

$$\mathcal{B}\dot{\omega} = [I\omega, \omega] - \left(\mathcal{B}^{-1}|_{h_0^g}\right)^{-1} \mathrm{pr}_{\mathfrak{h}_0^g} \mathcal{B}^{-1}([I\omega, \omega]), \tag{18}$$

$$D\dot{\mathbf{W}} = -\sum_{i=1}^{q} \frac{D}{\rho_i} \mathrm{pr}_{\mathfrak{h}_i}(\mathrm{Ad}_g \dot{\omega}), \tag{19}$$

$$\dot{g} = g \cdot \omega, \tag{20}$$

$$\dot{g}_1 = \mathbf{W} \cdot g_1. \tag{21}$$

where $\mathcal{B} = I + \Pi = I + \sum_{i=1}^{q} D\big/\rho_i^2\, \mathrm{pr}_{\mathfrak{h}_i^g}$ and $\mathcal{B}^{-1}|_{h_0^g} = \mathrm{pr}_{\mathfrak{h}_0^g} \circ \mathcal{B}^{-1} \circ \mathrm{pr}_{\mathfrak{h}_0^g} : \mathfrak{h}_0^g \to \mathfrak{h}_0^g$.

Proof. The equations of a motion in the right trivialization (or in the space frame) read

$$\dot{M} = \sum_{i=0}^{q} \Lambda_i, \tag{22}$$

$$D\dot{\mathbf{W}} = \sum_{i=1}^{q} \rho_i \Lambda_i, \tag{23}$$

$$\dot{g} = \Omega \cdot g, \tag{24}$$

$$\dot{g}_1 = \mathbf{W} \cdot g_1, \tag{25}$$

where the Lagrange multipliers (reaction forces) Λ_i belong to $\mathfrak{h}_i$ $(i = 0, 1, \ldots, q)$ and $M = \mathrm{Ad}_g(I\omega)$ is the first component of *angular momentum in the space frame* (the second component is $M_1 = D\mathbf{W}$).

Differentiating the constraints (16), from (23) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \Omega + \rho_i \mathbf{W}, \mathfrak{h}_i \rangle = \langle \dot{\Omega} + \rho_i \dot{\mathbf{W}}, \mathfrak{h}_i \rangle = \left\langle \dot{\Omega} + \rho_i \sum_{j=1}^{q} \frac{\rho_j}{D} \Lambda_j, \mathfrak{h}_i \right\rangle = \left\langle \dot{\Omega} + \frac{\rho_i^2}{D} \Lambda_i, \mathfrak{h}_i \right\rangle = 0,$$

that is

$$\Lambda_i = -\frac{D}{\rho_i^2} \mathrm{pr}_{\mathfrak{h}_i}(\dot{\Omega}), \qquad i = 1, \ldots, q. \tag{26}$$

Equation (19) follows from (23) and (26) and the relation

$$\dot{\Omega} = \mathrm{Ad}_g \dot{\omega}. \tag{27}$$

From (26) and identities (27), $\mathrm{pr}_{\mathfrak{h}_i^g} = \mathrm{Ad}_{g^{-1}} \mathrm{pr}_{\mathfrak{h}_i} \mathrm{Ad}_g$ and

$$\mathrm{Ad}_{g^{-1}} \dot{M} = I\dot{\omega} + [\omega, I\omega],$$

the equation (22) in the left trivialization takes the form

$$\mathcal{B}\dot{\omega} = [I\omega, \omega] + \lambda_0, \qquad \lambda_0 = \mathrm{Ad}_{g^{-1}}(\Lambda_0). \tag{28}$$

Now it remains to find the Lagrange multiplier λ_0. Differentiating (15) we get

$$\langle \dot{\Omega}, \mathfrak{h}_0 \rangle = \langle \mathrm{Ad}_g(\dot{\omega}), \mathfrak{h}_0 \rangle = \langle \dot{\omega}, \mathfrak{h}_0^g \rangle = 0.$$

By (28) it follows that $\lambda_0 = -\left(\mathcal{B}^{-1}|_{h_0^g}\right)^{-1} \mathrm{pr}_{\mathfrak{h}_0^g} \mathcal{B}^{-1}([I\omega, \omega])$. The proof is complete. $\qquad\square$

The Lagrangian (14) as well as constraints (16) are right ($\{\mathrm{Id}\} \times G_1$)-invariant and equations (18)–(20) can be seen as a reduction of the system to

$$\bar{\mathcal{D}} = \mathcal{D}/(\{\mathrm{Id}\} \times G_1) = \{(g, \omega, \mathbf{W})|\langle \mathrm{Ad}_g(\omega), \mathfrak{h}_0 \rangle = 0, \langle \mathrm{Ad}_g(\omega) + \rho_i \mathbf{W}, \mathfrak{h}_i \rangle = 0, i = 1, \ldots, q\}.$$

Let $\mathcal{D}_0 \subset TG$ be the right-invariant distribution defined by (15).

Theorem 3.2. *Equations (18)–(20) on $\bar{\mathcal{D}}$ reduce to the following system on $\mathcal{D}_0 \subset TG$:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}\omega) = [\mathcal{B}\omega, \omega] - \left(\mathcal{B}^{-1}|_{h_0^g}\right)^{-1} \mathrm{pr}_{\mathfrak{h}_0^g} \mathcal{B}^{-1}([I\omega, \omega]), \qquad \dot{g} = g \cdot \omega. \tag{29}$$

Proof. Equations (18) and (20) form a closed system on $\mathcal{D}_0$. If $(g(t), \omega(t))$ is a solution to (18) and (20), then one can easily reconstruct the motion of $\mathbf{W}$. Let

$$\mathfrak{k} = (\mathfrak{h}_1 + \cdots + \mathfrak{h}_q)^{\perp}. \tag{30}$$

From (20) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{pr}_{\mathfrak{k}} \mathbf{W} = 0, \tag{31}$$

while the $\mathfrak{h}_i$-components of the angular velocity $\mathbf{W}$ are determined from the constraints (16):

$$\mathrm{pr}_{\mathfrak{h}_i} \mathbf{W} = -1/\rho_i \, \mathrm{pr}_{\mathfrak{h}_i} \mathrm{Ad}_g(\omega), \qquad i = 1, \ldots, q.$$

Now, let $a_1, \ldots, a_{k_j}$ be the orthonormal base of $\mathfrak{h}_j$. Then $\alpha_1 = \mathrm{Ad}_{g^{-1}}(a_1), \ldots, \alpha_{k_j} = \mathrm{Ad}_{g^{-1}}(a_{k_j})$ will be the orthonormal base of $\mathfrak{h}_j^g$. We have

$$\mathrm{pr}_{\mathfrak{h}_j^g}(\omega) = \sum_{i=1}^{k_j} \alpha_i \otimes \alpha_i \omega = \sum_{i=1}^{k_j} \langle \alpha_i, \omega \rangle \alpha_i.$$

Whence, by using (7) and the identity $\langle \omega, [\alpha_i, \omega] \rangle = 0$, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{pr}_{\mathfrak{h}_j^g}(\omega)\right) = \sum_{i=1}^{k_j}(\langle \dot{\omega}, \alpha_i \rangle \alpha_i + \langle \omega, [\alpha_i, \omega] \rangle \alpha_i + \langle \omega, \alpha_i \rangle [\alpha_i, \omega])
$$

$$
= \mathrm{pr}_{\mathfrak{h}_j^g}(\dot{\omega}) + \sum_{i=1}^{k_j}\langle \omega, \alpha_i \rangle [\alpha_i, \omega].
$$

The above equation implies that (18) and (20) can be rewritten in the form (29). □

The derivation of $\langle \mathcal{B}\omega, \omega \rangle$ along the flow is as follows: $\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{B}\omega, \omega \rangle = 2\langle [\mathcal{B}, \omega], \omega \rangle + 2\langle \lambda_0, \omega \rangle$. The first term is equal to zero since $\langle \cdot, \cdot \rangle$ is an Ad_G-invariant scalar product, while the second term is equal to zero from the constraint (15). We can refer to $L_{\mathrm{red}} = \frac{1}{2}\langle \mathcal{B}\omega, \omega \rangle$ as to the *reduced Lagrangian*, or *reduced kinetic energy*. If $\mathrm{pr}_{\mathfrak{k}}\mathbf{W} \equiv 0$, the reduced kinetic energy coincides with the kinetic energy of the reconstructed motion on the whole phase space.

From equation (22) we also get the linear conservation law

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{pr}_{\mathfrak{k}_0}\mathrm{Ad}_g(I\omega)\right) = 0, \qquad \text{where} \quad \mathfrak{k}_0 = (\mathfrak{h}_0 + \mathfrak{h}_1 + \cdots + \mathfrak{h}_q)^{\perp}. \tag{32}
$$

The integrals (31) and (32) are actually Noether integrals (8) of the system. The other Noether integrals are trivial:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{pr}_{\mathfrak{h}_i}\mathrm{Ad}_g(I\omega) - \frac{D}{\rho_i}\mathrm{pr}_{\mathfrak{h}_i}\mathbf{W}\right) = 0, \qquad i = 1, \ldots, q.
$$

Remark 3.1. If $\mathfrak{h}_0 = 0$, i.e., we do not impose the constraint (15), the reduced system is an L+R system on the Lie group G

$$
\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}\omega) = [\mathcal{B}\omega, \omega], \qquad \dot{g} = g \cdot \omega. \tag{33}
$$

Further suppose that (30) is the Lie algebra of the closed Lie subgroup $K \subset G$ and that linear subspaces $\mathfrak{h}_i$ are Ad_K-invariant:

$$
\mathrm{Ad}_k\mathfrak{h}_i = \mathfrak{h}_i, \qquad k \in K, \qquad i = 1, \ldots, q.
$$

Then, since $\mathfrak{h}_i^{kg} = \mathfrak{h}_i^g, k \in K$, the L+R equations (33) are *left K-invariant* and we can reduce them to $G/K \times \mathfrak{g}$.

Remark 3.2. In the case when $\mathfrak{h}_0$ is the Lie algebra of a closed subgroup $H \subset G, \mathfrak{h}_1 + \mathfrak{h}_2 + \cdots + \mathfrak{h}_q = \mathfrak{g}$ and linear spaces $\mathfrak{h}_i$ are Ad_H invariant, then the coupled LR system (14)–(16) is a $(H \times G_1)$-Chaplygin system with respect to the action:

$$
(a, b) \cdot (g, g_1) = (ag, g_1b^{-1}), \qquad (a, b) \in H \times G_1.
$$

The reduced space $\mathcal{D}/(H \times G_1)$ is the tangent bundle of the homogeneous space G/H.

Theorem 3.3. *An arbitrary L+R system (11) can be seen as a reduction of an appropriate coupled LR system.*

Proof. Let $e_1, \ldots, e_n$ be the orthonormal base of $\mathfrak{g}$ in which the symmetric operator Π^0 has the diagonal form: $\Pi^0 = \sum_{i=1}^n \sigma_i e_i \otimes e_i$. Then the right-invariant term in (11) reads $\Pi = \Pi^g = \sum_{i=1}^n \sigma_i e_i^g \otimes e_i^g$, where e_i^g are given by

$$
e_1^g = \mathrm{Ad}_{g^{-1}}(e_1), \ldots, e_n^g = \mathrm{Ad}_{g^{-1}}(e_n). \tag{34}
$$

Consider the coupled nonholonomic LR system (14) and (16), where $q = n$ and $\mathfrak{h}_i$ are the lines in the directions of $e_i, i = 1, \ldots, n$. We can choose parameters D, ρ_i, such that $\sigma_i = D/\rho_i^2, i = 1, \ldots, n$. The system represents a $\{\mathrm{Id}\} \times G_1$- Chaplygin system with reduced equations of the required form (11). □

4. *N*-coupled systems

There is a straightforward generalization of the construction to the case when we have coupling with *N* different Lie groups, that is the configuration space is the direct product $G \times G_1 \times \cdots \times G_N$ and the Lagrangian is

$$L = \frac{1}{2}\langle I\omega, \omega\rangle + \frac{1}{2}\sum_{i=1}^{N} D_i \langle \mathbf{w}_i, \mathbf{w}_i\rangle_i, \tag{35}$$

where $\langle \cdot, \cdot \rangle_i$ are Ad_{G_i} invariant scalar products on Lie algebras $\mathfrak{g}_i = T_{\mathrm{Id}}G_i$, $i = 1, \ldots, N$.

Let us fix a base $e_1, \ldots, e_n$ of $\mathfrak{g}$ and some bases $f_1, \ldots, f_{d_i}$ of $\mathfrak{g}_i$ ($d_i = \dim \mathfrak{g}_i$). Let

$$A_i : \mathfrak{g} \to \mathbb{R}^{p_i}, \qquad B_i : \mathfrak{g}_i \to \mathbb{R}^{p_i}, \qquad i = 1, \ldots, N,$$

be the linear mappings with matrixes $[A_i]$ ($p_i \times n$) and $[B_i]$ ($p_i \times d_i$) in the above bases. In addition, we suppose that the ($p_i \times p_i$)-matrixes

$$[C_i] = [B_i][B_i]^T, \qquad i = 1, \ldots, N$$

are invertible. Consider the right-invariant constraints given by

$$A_i\Omega + B_i\mathbf{W}_i = 0, \qquad i = 1, \ldots, N. \tag{36}$$

Here, $\omega, \mathbf{w}_i$ and $\Omega, \mathbf{W}_i$ are velocities in the left and right trivializations, respectively and $D_i > 0$, $i = 1, \ldots, N$ are real parameters.

Let $[\Omega]$, $[\mathbf{W}_i]$ denote the column matrix, representing Ω and $\mathbf{W}_i$ in the chosen bases. We have $[\omega]_g = [\Omega]$, where $[\xi]_g$ is the column, representing $\xi \in \mathfrak{g}$ in the base (34).

In the right trivialization, the equation in $\mathbf{W}_i$ reads

$$D_i[\dot{\mathbf{W}}_i] = [B_i]^T[\lambda_i], \tag{37}$$

where $[\lambda_i]$ is the Lagrange multiplier ($p_i \times 1$)-matrix. Differentiating the constraints (36), from (37) we get

$$[\lambda_i] = -D[C_i]^{-1}[A_i][\dot{\Omega}], \qquad i = 1, \ldots, N. \tag{38}$$

Repeating the arguments of theorems 3.1 and 3.2, the considered *N*-coupled nonholonomic system reduces to the L+R system

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}\omega) = [\mathcal{B}\omega, \omega], \qquad \dot{g} = g \cdot \omega,$$

where $\mathcal{B}\omega = I\omega + \Pi\omega$ and $\Pi\omega$ in the matrix form, relative to the base (34), is given by

$$[\Pi\omega]_g = \sum_{i=1}^{N} D_i[A_i]^T[C_i]^{-1}[A_i][\omega]_g.$$

As above, one can easily incorporate an additional right-invariant constraint of the form (15).

4.1. LR systems on $G \times \mathfrak{g} \times \cdots \times \mathfrak{g}$

As an example, consider the case where G_i are all equal to the Lie algebra $\mathfrak{g}$ considered as an Abelian group, $\langle \cdot, \cdot \rangle_i = \langle \cdot, \cdot \rangle$ and the constraints (36) are given by

$$[\Gamma_i, \Omega] + \rho_i\mathbf{W}_i = 0, \qquad i = 1, \ldots, N, \tag{39}$$

where Γ_i are fixed elements of the Lie algebra $\mathfrak{g}$ and ρ_i are real parameters. Note that, since $G_i = \mathfrak{g}$ is an Abelian group, the angular velocities coincide with the usual velocity: $\dot{\xi}_i = \mathbf{W}_i = \mathbf{w}_i, \xi \in \mathfrak{g}$.

The equations of a motion in the right trivialization read

$$\dot{M} = \sum_{i=0}^{N} [\Lambda_i, \Gamma_i], \qquad \dot{g} = \Omega \cdot g, \tag{40}$$

$$D_i \dot{\mathbf{W}}_i = \rho_i \Lambda_i, \qquad \dot{\xi}_i = \mathbf{W}_i, \tag{41}$$

where $M = \mathrm{Ad}_g(I\omega)$. This is a $\{\mathrm{Id}\} \times \mathfrak{g}^N$–Chaplygin system and it is reducible to TG. Differentiating the constraints (39), from (41) we get the Lagrange multipliers

$$\Lambda_i = -\frac{D}{\rho_i^2} [\Gamma_i, \dot{\Omega}], \qquad i = 1, \ldots, N.$$

Therefore, equations (40) in the left trivialization take the form

$$I\dot{\omega} = [I\omega, \omega] - \sum_{i=1}^{N} \frac{D_i}{\rho_i^2} [[\gamma_i, \dot{\omega}], \gamma_i], \qquad \dot{g} = g \cdot \omega,$$

where $\gamma_i = \mathrm{Ad}_{g^{-1}}(\Gamma_i), i = 1, \ldots, N$. Next, from the identities

$$\frac{\mathrm{d}}{\mathrm{d}t} [[\gamma_i, \omega], \gamma_i] = [[\gamma_i, \dot{\omega}], \gamma_i] + [[[\gamma_i, \omega], \gamma_i], \omega], \qquad i = 1, \ldots, N,$$

we obtain the following proposition.

Proposition 4.1. *The reduced equations of the N-coupled nonholonomic system (35), (39) are given by the L+R system*

$$\frac{\mathrm{d}}{\mathrm{d}t} (\mathcal{B}\omega) = [\mathcal{B}\omega, \omega], \qquad \dot{g} = g \cdot \omega, \tag{42}$$

where

$$\mathcal{B}\omega = I\omega + \sum_{i=1}^{N} \frac{D_i}{\rho_i^2} [[\gamma_i, \omega], \gamma_i].$$

Remark 4.1. Nonholonomic systems on semi-direct products $G \times_\sigma V$, where σ is a representation of the Lie group G on the vector space V, are studied in Schneider [28]. Proposition 4.1 can be derived from theorem 3 given in [28].

5. Spherical support

Consider the motion of a dynamically nonsymmetric ball $\mathcal{S}$ in $\mathbb{R}^n$ with the unit radius around its fixed center. Suppose that the ball touches N arbitrary dynamically symmetric balls whose centers are also fixed, and there is no sliding at the contact points. We call this mechanical construction *the spherical support*. For $n = 3$ the spherical support is defined by Fedorov [13, 15].

The configuration space is $SO(n)^{N+1}$: the matrices $g, g_i \in SO(n)$ map the frames attached to the ball $\mathcal{S}$ and the ith peripheral ball to the fixed frame, respectively. The Lagrangian is of the form (35), where for $\langle \cdot, \cdot \rangle$ we take the scalar product proportional to the Killing form

$$\langle X, Y \rangle = -\tfrac{1}{2} \mathrm{tr}(XY), \tag{43}$$

the angular velocities $\omega, \Omega, \mathbf{w}_i, \mathbf{W}_i$ of the balls are defined as above, $I : so(n) \to so(n)$ is the inertia tensor of the ball $\mathcal{S}$ and $D_i, \rho_i \in \mathbb{R}$ are the central inertia moment and the radius of the ith peripheral ball.
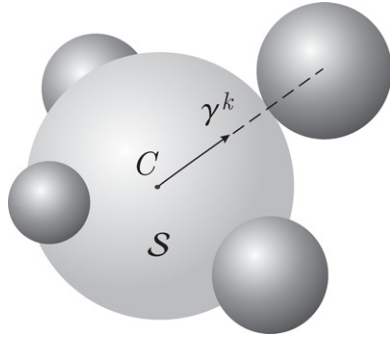
Figure 1. The spherical support.

Let $\Gamma_i \in \mathbb{R}^n$ be the unit vector *fixed in the space* and directed from the center C of the ball $\mathcal{S}$ to the point of contact with the ith ball. Nonholonomic constraints express the absence of sliding at the contact points. This means that the velocity of the point of contact of the ball $\mathcal{S}$ with the ith ball, in the space frame, is the same as the velocity of the corresponding point on the ith ball.

Consider the fixed point on the ball $\mathcal{S}$ with coordinates r and R in the body and space frames, respectively. Then the velocity of the point r in space is given by the Poisson equation (e.g., see [17]) $V = \dot{R} = \frac{\mathrm{d}}{\mathrm{d}t}(g \cdot r) = \dot{g} \cdot g^{-1} \cdot g \cdot r = \Omega R$. Therefore, the velocity of the contact point with the ith peripheral ball is given by $\Omega \Gamma_i$. Similarly, the velocity of the corresponding contact point of the ith ball in the space frame is given by $-\rho_i \mathbf{W} \Gamma_i$ and the constraints are

$$\Omega \Gamma_i + \rho_i \mathbf{W}_i \Gamma_i = 0, \qquad i = 1, \ldots, N. \tag{44}$$

We see that the n-dimensional spherical support is actually an N-coupled LR system studied in the previous section. Let

$$\gamma_i = g^{-1} \Gamma_i, \qquad i = 1, \ldots, N \tag{45}$$

be the contact points of $\mathcal{S}$ with the ith ball ($i = 1, \ldots, N$) in the frame attached to the ball $\mathcal{S}$. Then the right-invariant constraints (44) can be rewritten in the form

$$\langle \Omega + \rho_i \mathbf{W}_i, \mathfrak{h}_i \rangle = 0, \qquad i = 1, \ldots, N, \tag{46}$$

where

$$\mathfrak{h}_i = \mathbb{R}^n \wedge \Gamma_i, \qquad \mathfrak{h}_i^g = \mathrm{Ad}_{g^{-1}}(\mathfrak{h}_i) = \mathbb{R}^n \wedge \gamma_i, \qquad i = 1, \ldots, N$$

are linear (no mutually orthogonal) subspaces of the Lie algebra $so(n)$.

From the identity $\mathrm{pr}_{\mathfrak{h}_i^g} \dot{\omega} = (\dot{\omega} \gamma_i) \wedge \gamma_i = \dot{\omega} \gamma_i \otimes \gamma_i + \gamma_i \otimes \gamma_i \dot{\omega}$, the equations of the motion become

$$I\dot{\omega} = [I\omega, \omega] - \sum_{i=1}^N \frac{D_i}{\rho_i^2} (\dot{\omega} \gamma_i \otimes \gamma_i + \gamma_i \otimes \gamma_i \dot{\omega}), \qquad \dot{g} = g \cdot \omega,$$

$$D_i \dot{\mathbf{W}}_i = -\frac{D_i}{\rho_i} (\dot{\Omega} \Gamma_i \otimes \Gamma_i + \Gamma_i \otimes \Gamma_i \dot{\Omega}), \qquad \dot{g}_i = \mathbf{W}_i \cdot g_i, \quad i = 1, \ldots, N.$$

We have the conservation laws

$$\dot{\mathbf{W}}_i - \dot{\mathbf{W}}_i \Gamma_i \otimes \Gamma_i - \Gamma_i \otimes \Gamma_i \dot{\mathbf{W}}_i = 0, \qquad i = 1, \ldots, N,$$

which together with the right ($\{\mathrm{Id}\} \times SO(n)^N$)-symmetry lead to the following statement.

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Proposition 5.1. *The spherical support system reduces to the L+R flow*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}\omega) = [\mathcal{B}\omega, \omega], \qquad \dot{g} = g \cdot \omega, \tag{47}$$

where $\mathcal{B}\omega = I\omega + \sum_{i=1}^{N} D_i / \rho_i^2 \, (\omega\gamma_i \otimes \gamma_i + \gamma_i \otimes \gamma_i\omega)$ *and* γ_i *are defined by* (45).

Note that the vectors γ_i in the frame attached to the ball $\mathcal{S}$ satisfy the Poisson equations (e.g., see [17])

$$\dot{\gamma}_i = -\omega\gamma_i, \qquad i = 1, \ldots, N. \tag{48}$$

By introducing $\mathcal{X}_i = \gamma_i \otimes \gamma_i$, from (48) we obtain

$$\dot{\mathcal{X}}_i = [\mathcal{X}_i, \omega], \qquad i = 1, \ldots, N. \tag{49}$$

Combining (47) and (49) we get a family of integrals—the coefficients of the polynomials

$$\mathrm{tr}\left(\mathcal{B}\omega + \sum_{i=1}^{N} \mu^i \mathcal{X}_i\right)^k, \qquad k = 1, \ldots, n. \tag{50}$$

For $n = 3$ the system is integrable by the Euler–Jacobi theorem, and its generic invariant manifolds are two-dimensional tori (see [13, 15]).

Remark 5.1. If the positions of peripheral balls are mutually orthogonal

$$(\Gamma_i, \Gamma_j) = (\gamma_i, \gamma_j) = \delta_{ij}, \qquad 1 \leqslant i, \quad j \leqslant N \leqslant n,$$

then the components of γ_i can be seen as redundant coordinates on the Stiefel variety $V(n, N) = SO(n)/SO(n - N)$. The system is invariant with respect to the $SO(n - N)$ action, representing the rotations in the space orthogonal to $\mathrm{span}\{\gamma_1, \ldots, \gamma_N\}$. The $SO(n - N)$-reduced system on $TSO(n)/SO(n - N) \cong V(n, N) \times so(n)$ is given by Poisson equations (48) and the first equation in (47).

5.1. Rubber spherical support

Now consider the *rubber spherical support system* in $\mathbb{R}^n$. The analogue of rubber rolling is that, in addition to the constraints (46), the rotations of the ball $\mathcal{S}$ and ith peripheral ball around the vector Γ_i are the same:

$$\langle \Omega - \mathbf{W}_i, \mathfrak{k}_i \rangle = 0, \qquad i = 1, \ldots, N, \tag{51}$$

where

$$\mathfrak{k}_i = \mathfrak{h}_i^{\perp}, \qquad \mathfrak{k}_i \cong so(n - 1).$$

Since $\mathrm{pr}_{\mathfrak{k}_i} = \mathbf{I} - \mathrm{pr}_{\mathfrak{h}_i}$ we get

Proposition 5.2. *The rubber spherical support system is described by the equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}^*\omega) = [\mathcal{B}^*\omega, \omega], \tag{52}$$

$$D_i \dot{\mathbf{W}}_i = D_i \dot{\Omega} - D_i \frac{1 + \rho_i}{\rho_i} (\dot{\Omega}\Gamma_i \otimes \Gamma_i + \Gamma_i \otimes \Gamma_i\dot{\Omega}), \qquad i = 1, \ldots, N, \tag{53}$$

$$\dot{g} = g \cdot \omega, \tag{54}$$

$$\dot{g}_i = \mathbf{W}_i \cdot g_i, \qquad i = 1, \ldots, N, \tag{55}$$

where

$$\mathcal{B}^*\omega = I\omega + (D_1 + \cdots + D_N)\omega + \sum_{i=1}^{N} D_i \frac{1 - \rho_i^2}{\rho_i^2} (\omega\gamma_i \otimes \gamma_i + \gamma_i \otimes \gamma_i\omega).$$

Equations (53) are trivial since **W** can be expressed in terms of Ω from constraints (46) and (51).

As above, we get family of geometric integrals that can be expressed as the coefficients of the polynomials

$$\text{tr}\left(\mathcal{B}^*\omega + \sum_{i=1}^{N} \mu^i \mathcal{X}_i\right)^k, \qquad k = 1, \ldots, n. \tag{56}$$

For $n = 3$, among the reduced kinetic energy $\frac{1}{2}\langle \mathcal{B}^*\omega, \omega\rangle$ and integrals (56) there are four independent ones.

Theorem 5.3. *For $n = 3$, the rubber spherical support system (52) and (54) is solvable by the Euler–Jacobi theorem and its generic invariant manifolds are two-dimensional tori.*

6. Rubber Chaplygin sphere

Following [16, 17], consider the generalized Chaplygin sphere problem of an n-dimensional ball of radius ρ, rolling without slipping on an $(n - 1)$-dimensional hyperspace $\mathcal{H}$ in $\mathbb{R}^n$. For the configuration space we take the *direct product* of Lie groups $SO(n)$ and $\mathbb{R}^n$, where $g \in SO(n)$ is the rotation matrix of the sphere (mapping frame attached to the body to the space frame) and $r \in \mathbb{R}^n$ is the position vector of its center C (in the space frame). For a trajectory $(g(t), r(t))$ define angular velocities

$$\omega = g^{-1}\dot{g}, \qquad \Omega = \dot{g}g^{-1}, \qquad \mathbf{w} = \mathbf{W} = \dot{r}.$$

The Lagrangian of the system is then given by

$$L = \frac{1}{2}\langle I\omega, \omega\rangle + \frac{1}{2}m(\mathbf{w}, \mathbf{w}). \tag{57}$$

Here $I : so(n) \to so(n)$ and m are the inertia tensor and mass of the ball, $\langle \cdot, \cdot \rangle$ is given by (43) and $(\cdot, \cdot)$ is the Euclidean scalar product.

Let $\Gamma \in \mathbb{R}^n$ be a *vertical* unit vector (considered in the fixed frame) orthogonal to the hyperplane $\mathcal{H}$ and directed from $\mathcal{H}$ to the center C. The condition for the sphere to roll without slipping leads to the velocity of the contact point being equal to zero:

$$-\rho\Omega\Gamma + \mathbf{W} = 0. \tag{58}$$

This is a right-invariant nonholonomic constraint of the form (36). If we take the fixed orthonormal base $E_1 = (1, 0, \ldots, 0, 0)^T, \ldots, E_n = (0, 0, \ldots, 0, 1)^T$, such that $\Gamma = E_n$, then the constraint (58) takes the form

$$\dot{r}_i = \rho\Omega_{in}, \qquad i = 1, \ldots, n-1, \qquad \dot{r}_n = 0, \qquad \text{where} \quad \Omega_{ij} = \langle\Omega, E_i \wedge E_j\rangle.$$

The last constraint is holonomic, and for the physical motion we take $r_n = \rho$. From now on we take $SO(n) \times \mathbb{R}^{n-1}$ for the configuration space of the rolling sphere, where $\mathbb{R}^{n-1}$ is identified with the affine hyperplane $\rho\Gamma + \mathcal{H}$.

Let $\mathfrak{h} \subset so(n)$ be the linear subspace $\mathfrak{h} = \mathbb{R}^n \wedge \Gamma$ and $\mathfrak{k} \cong so(n-1)$ its orthogonal complement in $so(n)$. Define the *rubber Chaplygin sphere* as a Chaplygin sphere (57), (58) subjected to the additional right-invariant constraints

$$\langle\Omega, \mathfrak{k}\rangle = \langle\omega, \mathfrak{k}^g\rangle = 0, \qquad \mathfrak{k}^g = \text{Ad}_{g^{-1}}\mathfrak{k}, \quad \Longleftrightarrow \quad \Omega_{ij} = 0, \qquad 1 \leqslant i < j \leqslant n-1, \tag{59}$$

describing the no-twist condition at the contact point. As a result, the distribution

$$\mathcal{D} = \{(g, r, \omega, \mathbf{W}) | \langle \omega, \mathfrak{k}^g \rangle = 0, \mathbf{W} = \rho \mathrm{Ad}_g(\omega)\Gamma\}$$

is right $SO(n) \times \mathbb{R}^{n-1}$ as well as the left $SO(n-1) \times \mathbb{R}^{n-1}$ invariant ($SO(n-1)$ is the subgroup of $SO(n)$ with the Lie algebra $\mathfrak{k}$). Moreover, the rubber Chaplygin sphere is a $(SO(n-1) \times \mathbb{R}^{n-1})$-Chaplygin system.

Let $\gamma = g^{-1}\Gamma$ be the vertical vector in the frame attached to the ball. Then

$$\mathfrak{h}^g = \mathrm{Ad}_{g^{-1}}(\mathfrak{h}) = \mathbb{R}^n \wedge \gamma =: \mathfrak{h}^\gamma, \qquad \mathfrak{k}^g = \mathrm{Ad}_{g^{-1}}(\mathfrak{k}) = (\mathbb{R}^n \wedge \gamma)^\perp =: \mathfrak{k}^\gamma$$

and the reduced space $\mathcal{D}/(SO(n-1) \times \mathbb{R}^{n-1})$ is the tangent bundle TS^{n-1} of the sphere which can be identified by the position of γ.

The equations in the right trivialization read

$$\dot{M} = \Lambda_0 - \rho\Lambda_1 \wedge \Gamma, \qquad \dot{g} = \Omega \cdot g, \tag{60}$$

$$m\dot{\mathbf{W}} = \Lambda_1, \qquad\qquad \dot{r} = \mathbf{W}, \tag{61}$$

where $M = \mathrm{Ad}_g(I\omega)$ is the ball angular momentum in the space and $\Lambda_0 \in \mathfrak{h}$, $\Lambda_1 \in \mathbb{R}^n$ are Lagrange multipliers.

From (58) and (61) we find $\Lambda_1 = m\rho\dot{\Omega}\Gamma$. On the other hand

$$\Lambda_1 \wedge \Gamma = m\rho(\dot{\Omega}\Gamma) \wedge \Gamma = m\rho(\dot{\Omega}\Gamma \otimes \Gamma + \Gamma \otimes \Gamma\dot{\Omega}) = m\rho\,\mathrm{pr}_{\mathfrak{h}}(\dot{\Omega}). \tag{62}$$

Whence, we can write equations (60) as a closed system on $\mathcal{D}_0 \subset TSO(n)$, where $\mathcal{D}_0$ is the right-invariant distribution defined by (59) (reduction of $\mathbb{R}^{n-1}$-symmetry). From (27), (62) and the relation $\mathrm{pr}_{\mathfrak{h}^\gamma}(\dot{\omega}) = (\dot{\omega} \cdot \gamma) \wedge \gamma = \dot{\omega}\gamma \otimes \gamma + \gamma \otimes \gamma\dot{\omega}$, in the left trivialization of $TSO(n)$ the reduced system takes the form

$$I\dot{\omega} = [I\omega, \omega] - m\rho^2(\dot{\omega}\gamma \otimes \gamma + \gamma \otimes \gamma\dot{\omega}) + \lambda_0, \qquad \dot{g} = g \cdot \omega,$$

where $\lambda_0 = \mathrm{Ad}_{g^{-1}}(\Lambda_0)$. Let

$$\mathbf{k} = I\omega + m\rho^2\,\mathrm{pr}_{\mathfrak{h}^\gamma}\omega = I\omega + m\rho^2(\omega\gamma \otimes \gamma + \gamma \otimes \gamma\omega) \in so(n)^* \tag{63}$$

be the angular momentum of the ball relative to the contact point (see [17]). Then we have

Proposition 6.1. *The motion of the rubber Chaplygin sphere, in variables ω, g, is described by equations*

$$\dot{\mathbf{k}} = [\mathbf{k}, \omega] + \lambda_0, \qquad \dot{g} = g \cdot \omega, \tag{64}$$

or, in variables ω, γ, by equations

$$\dot{\mathbf{k}} = [\mathbf{k}, \omega] + \lambda_0, \qquad \dot{\gamma} = -\omega\gamma. \tag{65}$$

The Lagrange multiplier matrix λ_0 belongs to $\mathfrak{k}^\gamma$ and is determined from the constraints (59).

6.1. Reduction and Hamiltonization

From the constraints (59), the momentum (63) equals $\mathbf{k} = I\omega + m\rho^2\omega$. Therefore, as in the three-dimensional case [6, 11], equations (64) are equivalent to the motion of a rigid body about the fixed point with the left-invariant kinetic energy given by the inertia operator $I + m\rho^2\mathbf{I}$ and constraint (59) (n-dimensional Veselova rigid body problem [17, 30]).

Now we simply follow [18]. The reduced Lagrange–d'Alambert equations of the rubber Chaplygin sphere (57)–(59) on

$$TS^{n-1} \cong \mathcal{D}/(SO(n-1) \times \mathbb{R}^{n-1}) \cong \mathcal{D}_0/SO(n-1)$$

are given by

$$\left(\frac{\partial L_{\mathrm{red}}}{\partial \gamma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{\mathrm{red}}}{\partial \dot{\gamma}}, \xi\right) = \langle (I + m\rho^2)\Phi(\gamma, \dot{\gamma}), (\mathbf{I} - \mathrm{pr}_{\mathfrak{h}^\gamma})[\Phi(\gamma, \dot{\gamma}), \Phi(\gamma, \xi)]\rangle, \tag{66}$$

for all virtual displacements $\xi \in T_\gamma S^{n-1}$ (see [18]). Here $\Phi(\gamma, \dot{\gamma}) = \gamma \wedge \dot{\gamma}$ is the momentum mapping of the right $SO(n)$-action on the round sphere S^{n-1} and the reduced Lagrangian is given by

$$L_{\mathrm{red}}(\gamma, \dot{\gamma}) = \tfrac{1}{2}\langle (I + m\rho^2)\Phi(\gamma, \dot{\gamma}), \Phi(\gamma, \dot{\gamma})\rangle. \tag{67}$$

After the Legendre transformation

$$p = \frac{\partial L_{\mathrm{red}}}{\partial \dot{\gamma}} = \frac{\partial L_{\mathrm{reg}}}{\partial \Phi}\frac{\partial \Phi}{\partial \dot{\gamma}} = m\rho^2\dot{\gamma} - I\Phi \cdot \gamma$$

we can also write the reduced Lagrange–d'Alambert equations as a first-order system on the cotangent bundle T^*S^{n-1} which is realized as a subvariety of $\mathbb{R}^{2n} = (q, p)$ defined by constraints $(\gamma, \gamma) = 1$, $(\gamma, p) = 0$ (since $I\Phi$ is skew-symmetric, the momentum p satisfies $(\gamma, p) = 0$). The system takes the symmetric form

$$\dot{\gamma} = -\Phi(\gamma, \dot{\gamma}(\gamma, p))\gamma, \qquad \dot{p} = -\Phi(\gamma, \dot{\gamma}(\gamma, p))p, \tag{68}$$

where $\dot{\gamma} = \dot{\gamma}(\gamma, p)$ is the inverse of the Legendre transformation.

Let σ be the canonical volume $2(n-1)$-form on T^*S^{n-1}. Then we have (see [18])

Proposition 6.2. *The reduced system (68) on T^*S^{n-1} possesses an invariant measure*

$$1/\sqrt{\det(I + m\rho^2\mathbf{I}|_{\mathfrak{h}^\gamma})}\sigma, \qquad I + m\rho^2\mathbf{I}|_{\mathfrak{h}^\gamma} = \mathrm{pr}_{\mathfrak{h}^\gamma} \circ (I + m\rho^2\mathbf{I}) \circ \mathrm{pr}_{\mathfrak{h}^\gamma}.$$

Furthermore, as it follows from [18], with the operator I defined on the bi-vectors $X \wedge Y$ by a diagonal matrix $A = \mathrm{diag}(A_1, \ldots, A_n)$ by

$$I(X \wedge Y) = AX \wedge AY - m\rho^2 X \wedge Y, \tag{69}$$

the Chaplygin reducibility is applicable for any dimension.

Theorem 6.3.

  (i) *If the inertia operator is given by (69), the density of an invariant measure in proposition 6.2 takes the following simple form:*

$$(A\gamma, \gamma)^{-(n-2)/2}.$$

 (ii) *Under the time substitution $\mathrm{d}\tau = 1/\sqrt{(A\gamma, \gamma)}\,\mathrm{d}t$ the reduced system (66) (or (68)) becomes a Hamiltonian system describing a geodesic flow on S^{n-1} with the Lagrangian*

$$L^*\left(\gamma, \frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right) = \frac{1}{2}\left[\left(A\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}, \frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right)(A\gamma, \gamma) - \left(A\gamma, \frac{\mathrm{d}\gamma}{\mathrm{d}\tau}\right)^2\right]. \tag{70}$$

(iii) *For A with distinct eigenvalues, the latter system is algebraic completely integrable and generic invariant manifolds are $(n-1)$-dimensional tori.*

(iv) *Moreover, the $SO(n-1)$-reconstruction of the motion is solvable: the generic trajectories of the system (64) are straight lines (but not uniform) over $(n-1)$-dimensional invariant tori.*

The complete integration is presented in [18]. Given a solution $(g(t), \omega(t))$ of the system (64), the reconstruction of r-variable simply follows from the integration of the constraint (58)

$$r(t) - r(t_0) = \rho \int_{t_0}^{t} \text{Ad}_{g(t)} \omega(t) \Gamma \, \mathrm{d}t.$$

In the case $n = 3$, under the isomorphism between $so(3)$ and $\mathbb{R}^3$

$$\omega_{ij} = \varepsilon_{ijl} \omega_l, \qquad \mathbf{k}_{ij} = \varepsilon_{ijl} \mathbf{k}_l, \tag{71}$$

from (65) we obtain the classical rubber Chaplygin's ball equations [11]

$$\dot{\vec{\mathbf{k}}} = \vec{\mathbf{k}} \times \vec{\omega} + \lambda \vec{\gamma}, \qquad \dot{\vec{\gamma}} = \vec{\gamma} \times \vec{\omega}, \tag{72}$$

where λ is determined from the constraint $(\vec{\omega}, \vec{\gamma}) = 0$ and $\vec{\mathbf{k}} = I\vec{\omega} + m\rho^2\vec{\omega} - m\rho^2(\vec{\omega}, \vec{\gamma})\vec{\gamma} = I\vec{\omega} + m\rho^2\vec{\omega}$.

For $n = 3$, the relation (69) defines a generic inertia tensor. Thus the rubber Chaplygin sphere in $\mathbb{R}^3$ is integrable. Indeed, let $I : \mathbb{R}^3 \to \mathbb{R}^3$ be an arbitrary inertia tensor. Under the isomorphism (71), the matrix A is determined from (69) via

$$A = \Delta (I + m\rho^2 \mathbf{I})^{-1}, \qquad \Delta = \sqrt{\det(I + m\rho^2 \mathbf{I})}.$$

The Hamiltonizaton of the reduced system on T^*S^2 is obtained in [11, 12]. The Chaplygin multiplier given in theorem 6.3

$$\mathrm{d}\tau = \mathrm{d}t / \sqrt{\Delta((I + m\rho^2 \mathbf{I})^{-1}\gamma, \gamma)}$$

up to the multiplication by a constant, coincides with the expression obtained in [11, 12].

6.2. Remarks on the Chaplygin sphere

- Note that the Chaplygin sphere equations

$$\dot{\mathbf{k}} = [\mathbf{k}, \omega], \qquad \dot{\gamma} = -\omega\gamma, \qquad \mathbf{k} = I\omega + m\rho^2(\omega\gamma \otimes \gamma + \gamma \otimes \gamma\omega)$$

coincide with the reduced equations of the spherical support system for $N = 1$, where instead of D_1/ρ_1 we should put $m\rho^2$. This is not the case for rubber analogues of the systems.

- Borisov and Mamaev [5, 6] proved that the classical Chaplygin rolling sphere problem is Hamiltonian after an appropriate time rescaling. Recently, the Hamiltonization of the homogeneous Chaplygin rolling sphere problem in $\mathbb{R}^n$ is given in [20], while the Hamiltonization of the non-homogeneous reduced Chaplygin sphere problem is obtained in [25].

- Let us turn back to the coupled LR system described in proposition 4.1. Take $N = 1$ and denote $\Gamma_1 = \Gamma$, $\gamma_1 = \gamma$, $D_1 = m$, $\rho_1 = 1/\rho$. The system (42) is additionally G_Γ-invariant, where $G_\Gamma \subset G$ is the isotropy group of Γ. Let $\mathcal{O} = G/G_\Gamma$ be the adjoint orbit of Γ. Then (42) reduces to $\mathcal{O} \times \mathfrak{g}$

$$\dot{\mathbf{k}} = [\mathbf{k}, \omega], \qquad \dot{\gamma} = [\gamma, \omega], \qquad \mathbf{k} = \mathcal{B}\omega = I\omega + m\rho^2[[\gamma, \omega], \gamma]. \tag{73}$$

For $G = SO(3)$ we reobtain the equations of a motion of the Chaplygin sphere in $\mathbb{R}^3$. Thus the system (73) can be seen as an alternative generalization of the Chaplygin sphere problem.

Acknowledgments

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